Math 150, *Lecture Notes- Bonds* **Name_____________**

***Section 5.2** The Natural Logarithmic Function: Integration*

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\int \frac{1}{u}\,du = \ln|u| + C$

Ex.1 Using the Log Rule for Integration

Find $\int \frac{2}{x}\,dx = 2 \cdot \int \frac{1}{x}\,dx$

$= 2 \cdot \ln|x| + C$

$= 2\ln|x| + C$ ✓

Ex.2 Using the Log Rule with a Change of Variables

Find $\int \frac{1}{4x-1}\,dx.$

let $u = 4x - 1$

$\frac{du}{dx} = 4$

$\rightarrow du = \frac{du}{dx} \cdot dx$

$du = 4 \cdot dx$

$\frac{du}{4} = \frac{4 \cdot dx}{4}$

$\frac{du}{4} = dx$

$= \int \frac{1}{u}\left(\frac{du}{4}\right)$

$= \frac{1}{4} \cdot \int \frac{1}{u}\,du$

$= \frac{1}{4} \cdot \ln|u| + C$

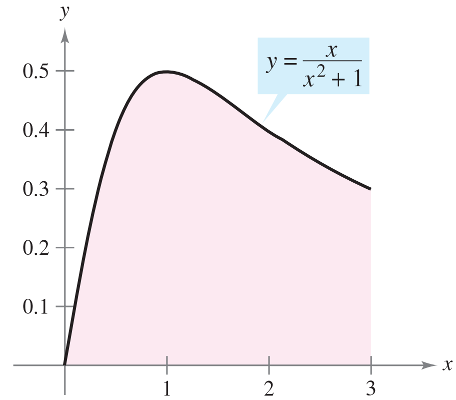
$= \frac{1}{4}\ln|4x-1| + C$ ✓

Ex.3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

the x-axis, and the line $x = 3$.



$$\text{Area} = \int_0^3 \frac{x}{x^2 + 1}\, dx$$

The area of the region bounded by the graph of y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.

Figure 5.8

$$\text{Area} = \int_0^3 \frac{x}{x^2+1}\,dx$$

$$= \int_{u=1}^{u=10} \frac{1}{u} \cdot \left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\int_1^{10} \frac{1}{u}\,du$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_1^{10}$$

$$= \frac{1}{2}\left[\ln(10) - \ln(1)\right]$$

$$= \frac{1}{2}\left[\ln(10) - 0\right]$$

$$= \frac{1}{2}\ln(10)$$

If $x = 0$
$u = (0)^2 + 1$
$u = 1$

If $x = 3$
$u = (3)^2 + 1$
$u = 9 + 1$
$u = 10$

Let $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

$$du = \frac{du}{dx} \cdot dx$$

$$du = 2x \cdot dx$$

$$\frac{du}{2} = \frac{2x \cdot dx}{2}$$

$$\frac{du}{2} = x \cdot dx$$

Ex.4 Recognizing Quotient Forms of the Log Rule

a. $\displaystyle\int \frac{3x^2+1}{x^3+x}\,dx = \int \frac{1}{u}\cdot du$

$= \ln|u| + C$

$= \ln|3x^2+1| + C$

Let

$u = x^3 + x$

$\frac{du}{dx} = 3x^2+1$

$du = \frac{du}{dx}\cdot dx$

$du = (3x^2+1)\cdot dx$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx = \int \frac{1}{u}\cdot du$

$= \ln|u| + C$

$= \ln|\sec^2(x)| + C$

$= \ln[\sec^2(x)] + C$

Let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$du = \frac{du}{dx}\cdot dx$

$du = \sec^2(x)\,dx$

c. $\displaystyle\int \frac{x+1}{x^2+2x}\,dx = \int \frac{1}{u}\left(\frac{du}{2}\right)$

$= \frac{1}{2}\cdot\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+2x| + C$

Let $u = x^2+2x$

$\frac{du}{dx} = 2x+2$

$du = \frac{du}{dx}\cdot dx$

$du = (2x+2)\cdot dx$

$\frac{du}{2} = \frac{2\cdot(x+1)\cdot dx}{2}$

$\frac{du}{2} = (x+1)\cdot dx$

d. $\displaystyle\int \frac{1}{3x+2}\,dx = \int \frac{1}{u}\left(\frac{du}{3}\right)$

$= \frac{1}{3}\cdot\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|3x+2| + C$

Let $u = 3x+2$

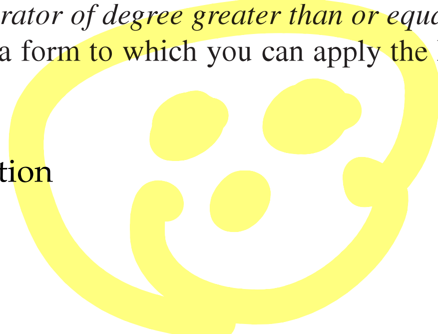
$\frac{du}{dx} = 3$

$du = \frac{du}{dx}\cdot dx$

$\frac{du}{3} = \frac{3\cdot dx}{3}$

$\frac{du}{3} = dx$

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.



Ex.5 Using Long Division Before Integration

Find $\displaystyle\int \frac{x^3 - 6x - 20}{x + 5}\,dx$

$$\begin{array}{r} x^2 - 5x + 19 \\ x+5 \,\big)\, x^3 + 0\cdot x^2 - 6x - 20 \\ -(x^3 + 5x^2) \\ \hline -5x^2 - 6x \\ -(-5x^2 - 25x) \\ \hline 19x - 20 \\ -(19x + 95) \\ \hline -115 \end{array}$$

① $x(x^2) = x^3$
② $x^2(x+5) = x^3 + 5x^2$

① $x(-5x) = -5x^2$
② $-5x(x+5) = -5x^2 - 25x$

① $x(19) = 19x$
② $19(x+5) = 19x + 95$

$$\frac{x^3 - 6x - 20}{x+5} = x^2 - 5x + 19 - \frac{115}{x+5}$$

let $u = x+5$

$\frac{du}{dx} = 1$

$du = \frac{du}{dx}\cdot dx$

$du = 1\cdot dx$

$du = dx$

$$\int \frac{x^3 - 6x - 20}{x+5}\,dx = \int \left[x^2 - 5x + 19 - \frac{115}{x+5}\right]dx$$

$$= \int x^2\,dx - 5\int x\,dx + 19\int dx - 115\int \frac{1}{x+5}\,dx$$

$$= \frac{x^3}{3} - 5\cdot\frac{x^2}{2} + 19x - 115\int \frac{1}{u}\,du$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 19x - 115\ln|u| + C$$

$$= \frac{x^3}{3} - \frac{5}{2}x^2 + 19x - 115\ln|x+5| + C$$

Ex.6 Change of Variables with the Log Rule

Find $\int \frac{2x}{(x+1)^2}\,dx.$

$$= \int \frac{2(u-1)}{u^2}\,du$$

$$= 2\int \frac{u-1}{u^2}\,du$$

$$= 2\int \left[\frac{u}{u^2} - \frac{1}{u^2}\right] du$$

$$= 2\int \left[\frac{1}{u} - \frac{1}{u^2}\right] du$$

$$= 2\int \frac{1}{u}\,du - 2\int \frac{1}{u^2}\,du$$

$$= 2\ln|u| - 2\int u^{-2}\,du$$

$$= 2\ln|u| - 2\left[\frac{u^{-1}}{-1}\right] + C$$

$$= 2\ln|u| + 2u^{-1} + C$$

$$= 2\ln|u| + \frac{2}{u} + C$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C \checkmark$$

let $u = x+1$

$$\frac{du}{dx} = 1$$

$$du = \frac{du}{dx}\cdot dx$$

$$du = 1\cdot dx$$

$$du = dx$$

If $u = x+1$,

then $u-1 = x+1-1$

$$u-1 = x$$

Guidelines for Integration

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, or addition and subtraction of the same quantity. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

$$dy = \frac{dy}{dx} \cdot dx$$

Ex.7 *u*-Substitution and the Log Rule

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

$$\int \left(\frac{dy}{dx}\right) dx = \int \left[\frac{1}{x \ln(x)}\right] dx$$

$$\int dy = \int \frac{1}{u} \cdot du$$

$$y = \ln|u| + C$$

$$y = \ln|\ln(x)| + C$$

let $u = \ln(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{du}{dx} \cdot dx$$

$$du = \frac{1}{x} \cdot dx$$

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Ex.8 Using a Trigonometric Identity

Find $\displaystyle\int \tan x \, dx$.

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)} dx$$

$$= \int \frac{1}{u} \cdot (-du)$$

$$= -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

Let $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$du = \frac{du}{dx} \cdot dx$$

$$du = -\sin(x) \cdot dx$$

$$-du = \sin(x) \cdot dx$$

We know: $\int \sec^2(x)\,dx = \tan(x) + C$

$\int \sec(x)\tan(x)\,dx = \sec(x) + C$

Ex.9 Derivation of the Secant Formula

Find $\displaystyle\int \sec x\,dx.$

$$\int \sec(x)\,dx = \int \frac{\sec(x)}{1} \cdot \left[\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right] dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx$$

$$= \int \frac{1}{u} \cdot du$$

$$= \ln|u| + C$$

$$= \ln|\sec(x) + \tan(x)| + C$$

Let $u = \sec(x) + \tan(x)$

$$\frac{du}{dx} = \sec(x)\tan(x) + \sec^2(x)$$

$$du = \frac{du}{dx} \cdot dx$$

$$du = [\sec(x)\tan(x) + \sec^2(x)] \cdot dx$$

$$du = [\sec^2(x) + \sec(x)\tan(x)]\,dx$$

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Ex.10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx.$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec(x)\, dx$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$= \Big[\ln|\sec(x) + \tan(x)|\Big]_0^{\frac{\pi}{4}}$$

on $[0, \frac{\pi}{4}]$

$$\sqrt{\sec^2(x)} = \sec(x) \geq 0$$

$$= \ln\left|\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \ln(\sqrt{2} + 1) - \ln(1)$$

$$= \ln(\sqrt{2} + 1) - 0$$

$$= \ln(\sqrt{2} + 1)$$

$$\sec\left(\tfrac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1}$$

Ex.11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

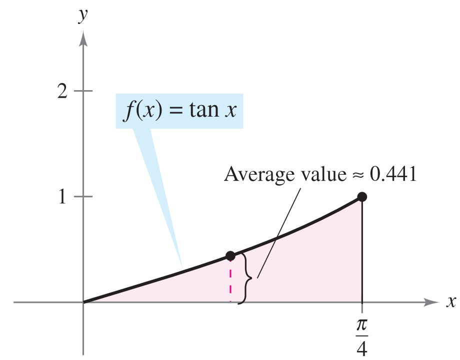


Figure 5.9

Average value on $[0, \frac{\pi}{4}]$ $= \dfrac{1}{\frac{\pi}{4} - 0} \displaystyle\int_0^{\frac{\pi}{4}} \tan(x)\,dx$

$$= \frac{1}{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \tan(x)\,dx$$

$$= \frac{4}{\pi}\Big[-\ln|\cos(x)|\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{-4}{\pi}\Big[\ln|\cos(x)|\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{-4}{\pi}\left[\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - \ln|\cos(0)|\right]$$

$$= \frac{-4}{\pi}\left[\ln\left|\frac{\sqrt{2}}{2}\right| - \ln|1|\right]$$

$$= \frac{-4}{\pi}\left[\ln\sqrt{2} - \ln(2) - 0\right]$$

$$= \frac{-4}{\pi}\left[\ln(2^{1/2}) - \ln(2)\right]$$

$$= \frac{-4}{\pi}\left[\frac{1}{2}\ln(2) - \frac{2}{2}\ln(2)\right]$$

$$= \frac{-4}{\pi}\left[-\frac{1}{2}\ln(2)\right] = \frac{2}{\pi}\ln(2)$$